

The isotropy constant and boundary properties of convex bodies *

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Abstract

Let \mathcal{K}^n be the set of all convex bodies in \mathbb{R}^n endowed with the Hausdorff distance. We prove that if $K \in \mathcal{K}^n$ has positive generalized Gauss curvature at some point of its boundary, then K is not a local maximizer for the isotropy constant L_K .

1 Introduction and statement of the main result.

Let K be a convex body in \mathbb{R}^n endowed with its canonical scalar product and Euclidean norm denoted by $|\cdot|$. It is well known (as a standard reference to the subject we refer to [BGVV]; another, earlier, comprehensive reference is [MP]) that there exists a unique (up to orthogonal transformations) affine, volume preserving, mapping $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for some constant $M_K > 0$, depending on K , one has for every $y \in \mathbb{R}^n$

$$\int_{AK} \langle x, y \rangle dx = 0 \text{ and } \int_{AK} \langle x, y \rangle^2 dx = M_K^2 |y|^2.$$

We say that K is *in isotropic position* (or that K is *isotropic*) if A is the identity on \mathbb{R}^n . The *isotropy constant* L_K of K is defined by

$$L_K = \frac{M_K}{|K|^{\frac{n+2}{2n}}}.$$

where $|B|$ denotes the volume of a Borel subset B of \mathbb{R}^n . Note that it is customary to assume, as part of the definition of isotropic position, that $|AK| = 1$; for the sake of convenience in our proofs, we prefer not to include this assumption in the definition.

The famous *Slicing Problem* asks whether there exists a universal constant $C > 0$ such that, for any n , any convex body K in \mathbb{R}^n has a hyperplane section $K \cap H$ such that

$$\text{vol}_{n-1}(K \cap H) \geq C \text{vol}_n(K)^{\frac{n-1}{n}}.$$

This problem is equivalent to the existence of an upper bound $D > 0$ for L_K , independent of the dimension. J. Bourgain proved in [B] that $L_K \leq Cn^{1/4} \log(n)$, this bound was improved by B. Klartag in [K] to $L_K \leq Cn^{1/4}$, where C is an absolute constant. Note that

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the minimum of L_K is obtained only for ellipsoids (for an interesting discussion of stability in that inequality, see [AB]).

Since the exact upper bound for L_K is still an open problem, it is interesting to investigate what are the properties of the maximizers for this quantity (a compactness argument shows that, for a fixed n , maximizers for L_K exist among convex bodies in \mathbb{R}^n). We say that a convex body K in \mathbb{R}^n is a *local maximizer* (resp. *local minimizer*) for L_K if for some $\varepsilon > 0$ one has $L_{K'} \leq L_K$ (resp. $L_{K'} \geq L_K$) for all convex bodies K' in \mathbb{R}^n such that $d(K', K) < \varepsilon$ (d may denote here the Hausdorff or the Banach–Mazur distance). L. Rademacher proved in [R] that if a simplicial polytope is a maximizer for L_K , then it must be a simplex. Campi, Colesanti and Gronchi showed in [CCG], using shadow movements, that if K has an open subset of its boundary which is C^2 with positive Gauss curvature, then K can not be a (local) maximizer of L_K in \mathbb{R}^n .

The main result of this paper is the following strong version of the result of [CCG]:

Theorem 1. *If a convex body K in \mathbb{R}^n is a local maximizer for L_K , then it has no positive generalized Gauss curvature at any point of its boundary. The same is true for a centrally symmetric K which is a local maximizer for L_K among centrally symmetric convex bodies.*

An open problem is whether a maximizer for L_K is necessarily a polytope. Our result is a step in this direction, because it shows that a maximizer has generalized Gauss curvature equal to 0 almost everywhere and never positive on its boundary. To prove theorem 1, we shall suppose that a convex body K has a positive generalized curvature at some point X_0 of its boundary (see Definition 1 below), modify slightly K in a neighborhood of X_0 , from inside and from outside to get a body K' for which we shall estimate $L_{K'}$. The paper is organized as follows. In section 2, after presenting some notations, we study the effect of such modifications, that are described in the general case in Lemmas 1, 2 and 3, and in the neighborhood of some special points of the boundary of K in Proposition 4 and Lemma 5. Corollary 6 is a generalization of [CCG]’s result, replacing positive curvature by strict convexity on an open subset of the boundary. To estimate carefully the asymptotic behavior of $L_{K'}$, we prove the geometric Lemma 7 and we get in Lemma 8 a special property of potential maximizers of L_K . Finally section 3 is devoted to the proof of theorem 1, which needs some technical and very precise computations of volumes.

In connection to Theorem 1, one should mention the paper [RSW], by Reisner, Schütt and Werner, where an analogous result is proved related to Mahler’s conjecture. Namely: a minimizer K of the volume-product can not have a point of positive generalized Gauss curvature on its boundary (see also [GM]).

2 Notations and preliminary results.

Let K be a convex body in \mathbb{R}^n . It is not hard to show, and is well known, that for any convex body K , denoting by $g(K)$ the centroid of K , one has

$$\begin{aligned} M_K^{2n} &= \frac{1}{n!} \int_{K-g(K)} \dots \int_{K-g(K)} (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n \\ &= \frac{1}{n!} \int_K \dots \int_K (\det(X_1 - g(K), \dots, X_n - g(K)))^2 dX_1 \dots dX_n. \end{aligned}$$

Let $X_0 \in \partial K$. For $r > 0$, denote $B(X_0, r)$ the Euclidean ball of center X_0 and radius r .

Definition 1. We say that K has positive generalized (Gauss) curvature at X_0 , if there exists an inner normal N of K at X_0 and a positive definite quadratic form q on $N^\perp = \{x \in \mathbb{R}^n; \langle x, N \rangle = 0\}$ such that for every $\varepsilon > 0$, there exists $a > 0$, such that whenever $Y \in N^\perp$ and $y \in \mathbb{R}$ satisfy

$$X_0 + Y + yN \in \partial K \cap B(X_0, a),$$

then

$$(1 - \varepsilon)q(Y) \leq y \leq (1 + \varepsilon)q(Y).$$

Of course, this normal N and the quadratic form q are then unique. Observe that if K is C^2 with positive curvature, then K has positive generalized curvature at any point X of its boundary, but that positive generalized curvature at some point X_0 does not imply any regularity at any point of ∂K other than X_0 . We refer to [SW] for more details on positive generalized curvature.

The following two lemmas show the effect of local slight modifications of an isotropic body K on $\int_K \dots \int_K (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n$.

Lemma 1. Let K be an isotropic convex body. Suppose that $C_m, m \geq 1$ is a sequence of Borel subsets of \mathbb{R}^n such that $C_m \cap \text{int}(K) = \emptyset$, $|C_m| > 0$, $|C_m| \rightarrow 0$ and $K_m := K \cup C_m$ is a convex body. Then, when $m \rightarrow +\infty$,

$$\frac{1}{n!} \int_{K_m} \dots \int_{K_m} (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n = M_K^{2n} + M_K^{2(n-1)} \int_{C_m} |X|^2 dX + O(|K_m \setminus K|^2).$$

Lemma 2. Let K be an isotropic convex body. Suppose that $D_m, m \geq 1$ is a sequence of Borel subsets of \mathbb{R}^n such that $D_m \subset K$, $|D_m| > 0$, $|D_m| \rightarrow 0$ and $K'_m := K \setminus D_m$ is a convex body. Then, when $m \rightarrow +\infty$,

$$\frac{1}{n!} \int_{K'_m} \dots \int_{K'_m} (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n = M_K^{2n} - M_K^{2(n-1)} \int_{D_m} |X|^2 dX + O(|K \setminus K'_m|^2).$$

Proof of Lemma 1 and Lemma 2:

One has

$$\begin{aligned} & \frac{1}{n!} \int_{K_m} \dots \int_{K_m} (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n \\ &= \frac{1}{n!} \left(\int_K \dots \int_K (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n \right. \\ & \quad \left. + n \int_{C_m} \int_K \dots \int_K (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n + O(|K_m \setminus K|^2) \right). \end{aligned}$$

Now

$$\int_{C_m} \int_K \dots \int_K (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n$$

$$= \int_{C_m} \int_K \cdots \int_K \left(\sum_{\sigma \in S_n} \sum_{\tau \in S_n} (-1)^{\varepsilon(\sigma)\varepsilon(\tau)} \prod_{i=1}^n X_{i\sigma(i)} X_{i\tau(i)} \right) dX_1 \dots dX_n.$$

Since K is isotropic one has

$$\int_K X_{i\sigma(i)} X_{i\tau(i)} dX_i = 0 \text{ if } \sigma(i) \neq \tau(i). \quad (1)$$

It follows that

$$\begin{aligned} & \int_{C_m} \int_K \cdots \int_K (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n \\ &= \int_{C_m} \int_K \cdots \int_K \left(\sum_{\sigma \in S_n} \sum_{\tau \in S_n} (-1)^{\varepsilon(\sigma)\varepsilon(\tau)} \prod_{i=1}^n X_{i\sigma(i)} X_{i\tau(i)} \right) dX_1 \dots dX_n \\ &= \sum_{\sigma \in S_n} \int_{C_m} \int_K \cdots \int_K \prod_{i=1}^n X_{i\sigma(i)}^2 dX_1 \dots dX_n \\ &= (n-1)! M_K^{2(n-1)} \int_{C_m} \sum_{m=1}^n X_{1m}^2 dX_1 \\ &= (n-1)! M_K^{2(n-1)} \int_{C_m} |X|^2 dX. \end{aligned}$$

We can thus conclude. The proof of lemma 2 is analogous. \square

In the next lemma, we investigate, under the hypotheses of lemmas 1 and 2 how M_{K_m} differ from M_K .

Lemma 3. *Under the hypotheses of Lemma 1 or respectively of Lemma 2, one has*

$$M_{K_m}^{2n} = \frac{1}{n!} \int_{K_m} \cdots \int_{K_m} (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n + O(|K_m \setminus K|^2)$$

or respectively,

$$M_{K'_m}^{2n} = \frac{1}{n!} \int_{K'_m} \cdots \int_{K'_m} (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n + O(|K \setminus K'_m|^2).$$

Proof: We assume throughout the proof that K is isotropic but, a posteriori, the equalities stated in the lemma remain true under invertible linear transformations.

Let g_m be the centroid of K_m . One has :

$$M_{K_m}^{2n} = \frac{1}{n!} \int_{K_m - g_m} \cdots \int_{K_m - g_m} (\det(X_1, \dots, X_n))^2 dX_1 \dots dX_n.$$

Since the centroid of K is at 0. One has for every $u \in S^{n-1}$,

$$\langle g_m, u \rangle = \frac{1}{|K| + |C_m|} \left(\int_K \langle X, u \rangle dX + \int_{C_m} \langle X, u \rangle dX \right) = \frac{1}{|K| + |C_m|} \int_{C_m} \langle X, u \rangle dX,$$

and thus $|g_m| = O(|C_m|)$ (observe that the hypotheses imply that the C_m , $m \geq 1$, are uniformly bounded).

We have

$$\begin{aligned}
n! M_{K_m}^{2n} &= \int_{K_m} \dots \int_{K_m} (\det(Y_1 - g_m, \dots, Y_n - g_m))^2 dY_1 \dots dY_n \\
&= \int_{K_m} \dots \int_{K_m} \left(\det(Y_1, \dots, Y_m) - \sum_{k=1}^n \det(Y_1, \dots, Y_{k-1}, g_m, Y_{k+1}, \dots, Y_n) \right)^2 dY_1 \dots dY_n \\
&= \mathbf{A} - \mathbf{B} + \mathbf{C}.
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{A} &:= \int_{K_m} \dots \int_{K_m} (\det(Y_1, \dots, Y_n))^2 dY_1 \dots dY_n \\
\mathbf{B} &:= 2 \sum_{k=1}^n \int_{K_m} \dots \int_{K_m} \det(Y_1, \dots, Y_n) \det(Y_1, \dots, Y_{k-1}, g_m, Y_{k+1}, \dots, Y_n) dY_1 \dots dY_n \\
\mathbf{C} &:= \int_{K_m} \dots \int_{K_m} \left(\sum_{k=1}^n \det(Y_1, \dots, Y_{k-1}, g_m, Y_{k+1}, \dots, Y_n) \right)^2 dY_1 \dots dY_n
\end{aligned}$$

The term \mathbf{A} has been treated already :

$$\frac{\mathbf{A}}{n!} = M_K^{2n} + M_K^{2(n-1)} \int_{C_m} |X|^2 dX + O(|C_m|^2).$$

Since $|g_m| = O(|C_m|)$, it is clear that

$$\mathbf{C} = O(|C_m|^2).$$

For \mathbf{B} we write

$$\frac{\mathbf{B}}{2} = \mathbf{D} + \mathbf{E} + O(|C_m|^2)$$

where

$$\mathbf{D} := \sum_{k=1}^n \int_K \dots \int_K \det(Y_1, \dots, Y_n) \det(Y_1, \dots, Y_{k-1}, g_m, Y_{k+1}, \dots, Y_n) dY_1 \dots dY_n$$

and

$$\mathbf{E} := \sum_{k=1}^n \int_K \dots \int_K \int_{C_m} \int_K \dots \int_K \det(Y_1, \dots, Y_n) \det(Y_1, \dots, Y_{k-1}, g_m, Y_{k+1}, \dots, Y_n) dY_1 \dots dY_n.$$

It is easily seen that $\mathbf{D} = 0$, because of the isotropicity of K . Now, once again since $g_m = O(|C_m|)$, one has $\mathbf{E} = O(|C_m|^2)$.

The corresponding result for K'_m is proved in the same way. \square

Proposition 4. *Under the assumptions of Lemma 1 or, respectively, Lemma 2 one has*

$$L_{K_m}^{2n} = L_K^{2n} \left[1 + \frac{\int_{K_m \setminus K} |X|^2 dX}{M_K^2} - (n+2) \frac{|K_m \setminus K|}{|K|} + O(|K_m \setminus K|^2) \right] \quad (2)$$

or, respectively,

$$L_{K'_m}^{2n} = L_K^{2n} \left[1 - \frac{\int_{K \setminus K'_m} |X|^2 dX}{M_K^2} + (n+2) \frac{|K \setminus K'_m|}{|K|} + O(|K \setminus K'_m|^2) \right]. \quad (3)$$

Proof: By Lemma 1 and Lemma 3 we have

$$L_{K_m}^{2n} = \frac{M_K^{2n} + M_K^{2(n-1)} \int_{K_m \setminus K} |X|^2 dX + O(|K_m \setminus K|^2)}{|K|^{n+2} + (n+2)|K|^{n+1}|K_m \setminus K| + O(|K_m \setminus K|^2)}.$$

From this (2) follows. The equality (3) is proved in a similar way. \square

Lemma 5. *Suppose that K is an isotropic convex body and that, in addition to the conditions of Proposition 4, there exists $X_0 \in \partial K$ such that X_0 is in the closure of C_m for all m and $\text{diam}(C_m) \rightarrow 0$ and also, X_0 is in the closure of D_m for all m and $\text{diam}(D_m) \rightarrow 0$. Then, if K is a local maximizer or a local minimizer for L_K , we have*

$$|X_0|^2 |K| = (n+2) M_K^2. \quad (4)$$

Proof: The conditions of the lemma imply that, when $m \rightarrow +\infty$, one has:

$$\int_{K_m \setminus K} |X|^2 dX \sim |X_0|^2 |K_m \setminus K| \text{ and } \int_{K \setminus K'_m} |X|^2 dX \sim |X_0|^2 |K \setminus K'_m|. \quad (5)$$

thus the result follows from Proposition 4. \square

Remarks

1) A common example of a point X_0 that satisfies the assumptions of Lemma 5 is the following: Let $X_0 \in \partial K$. We say that ∂K is *locally strictly convex* at X_0 or that X_0 is a *point of local strict convexity* of ∂K , if there exists no non-degenerate line segment $I \subset \partial K$ such that $X_0 \in I$ (even as an end-point). The following claim is easy to prove:

Claim. *Let X_0 be a point of local strict convexity of ∂K and let $N \in S^{n-1}$ be an outer normal of K at X_0 . Then the sets*

$$C_m = \text{conv} \left(K \cup \left(X_0 + \frac{1}{m} N \right) \right) \setminus K$$

and

$$D_m = \left\{ X \in K; \langle X, N \rangle \geq \langle X_0, N \rangle - \frac{1}{m} \right\}$$

satisfy the conditions of Lemma 5.

2) If $X_0 \in \partial K$ is a point of positive generalized curvature of ∂K then it is a point of local strict convexity and thus satisfies the conditions of Lemma 5.

As a corollary of Lemma 5 and of [CCG] (or of our Theorem 1) we get the following strengthening of a result of [CCG]:

Corollary 6. *Suppose that there exists an open neighborhood U in ∂K which is strictly convex (that is, every point in U is a point of local strict convexity). Then K is not a local maximizer for L_K .*

Proof: We may assume that K is isotropic. By Lemma 5 and the Claim following it, all the points in U have the same Euclidean norm. Thus U is an open neighborhood on a Euclidean sphere. The result of [CCG] or Theorem 1 now complete the proof. \square

We shall later need the following geometric lemma.

Lemma 7. *Suppose that K is a convex body containing 0 in its interior and that ∂K has positive generalized curvature at some point X_0 . Assume that the normal vector of K at X_0 is not parallel to the vector X_0 . Then there exists $u \in S^{n-1}$ and $\alpha > 0$ such that if $K(\alpha, u) = \{X \in K; \langle X, u \rangle \geq \alpha\}$, then $K(\alpha, u)$ is a cap of K with non-empty interior and*

$$\max_{X \in K(\alpha, u)} |X| < |X_0|.$$

Proof: After an affine change of variables in \mathbb{R}^n , transforming 0 into X_0 , we may suppose that for $|Z| \leq a$, the boundary of K is described by $z = g(Z)$ with $(Z, z) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, and

$$(1 - \varepsilon)|Z|^2 \leq g(Z) \leq (1 + \varepsilon)|Z|^2.$$

This affine change of variables transforms $B(0, |X_0|)$ into an ellipsoid \mathcal{E} with $0 \in \partial \mathcal{E}$, whose inner normal N at 0 is not e_n . We may suppose that $N = \cos(\theta)e_1 + \sin(\theta)e_n$ for some angle $\theta \in [0, \frac{\pi}{2}]$. Also, since \mathcal{E} has positive curvature at 0, one can find some positive constants b and C such that

$$B(0, b) \cap \mathcal{P} \subset B(0, b) \cap \mathcal{E} \quad (6)$$

where \mathcal{P} is the paraboloid defined by

$$\mathcal{P} = \{M := xe_1 + Y + ze_n; \langle OM, N \rangle \geq C(|OM|^2 - \langle OM, N \rangle^2)\}.$$

Let $0 < x_0 < a$. The hyperplane H tangent to the upper paraboloid ($z = (1 + \varepsilon)|Z|^2$) at $M_0 = x_0e_1 + (1 + \varepsilon)|x_0|^2e_n$ has the equation

$$z = (1 + \varepsilon)(2xx_0 - x_0^2),$$

where $M = xe_1 + Y + ze_n$ is a point in \mathbb{R}^n , with $Y \in \{e_1, e_n\}^\perp$. The zone \mathcal{A} between the hyperplane H and the lower paraboloid ($z = (1 - \varepsilon)|Z|^2$) is described by

$$\mathcal{A} = \{M : xe_1 + Y + ze_n; (1 - \varepsilon)(x^2 + |Y|^2) \leq z \leq (1 + \varepsilon)(2xx_0 - x_0^2)\}$$

Thus for $M \in \mathcal{A}$, one has

$$x^2 - 2\frac{1 + \varepsilon}{1 - \varepsilon}x_0x + \frac{1 + \varepsilon}{1 - \varepsilon}x_0^2 \leq 0$$

which says that

$$\left(x - \frac{1 + \varepsilon}{1 - \varepsilon}x_0\right)^2 \leq \frac{1 + \varepsilon}{1 - \varepsilon}\left(\frac{1 + \varepsilon}{1 - \varepsilon} - 1\right)x_0^2$$

or

$$\left(\frac{1 + \varepsilon}{1 - \varepsilon} - \frac{\sqrt{2\varepsilon(1 + \varepsilon)}}{1 - \varepsilon}\right)x_0 \leq x \leq \left(\frac{1 + \varepsilon}{1 - \varepsilon} + \frac{\sqrt{2\varepsilon(1 + \varepsilon)}}{1 - \varepsilon}\right)x_0.$$

It follows that for ε small enough one has for $M = xe_1 + Y + ze_n \in \mathcal{A}$: $x < 2x_0$ and $x^2 + |Y|^2 \leq 3x_0^2$. Thus, for x_0 small enough, $\mathcal{A} \cap \{xe_1 + Y + ze_n; z \geq g(x, Y)\}$ is a cap of K . passing through 0, with normal $N = \cos(\theta)e_1 + \sin(\theta)e_n$.

By (6), it is sufficient to show that for x_0 small enough, one has

$$\mathcal{A} \subset \mathcal{P} \cap B(0, b).$$

First it is easy to choose x_0 small enough such that $\mathcal{A} \subset B(0, b)$ Observe then that

$$\mathcal{P} = \{xe_1 + Y + ze_n; x \cos(\theta) + z \sin(\theta) \geq C(x^2 + |Y|^2 - (x \cos(\theta) + z \sin(\theta))^2)\}$$

and that setting $x = x_0 u$, $Y = x_0 V$ and $z = x_0^2 w$, one gets

$$\mathcal{A} = \{x_0(u + V + x_0 w); (1 - \varepsilon)(u^2 + |V|^2) \leq w \leq (1 + \varepsilon)(2u - 1)\}$$

Thus we need only to prove that if $(1 - \varepsilon)(u^2 + |V|^2) \leq w \leq (1 + \varepsilon)(2u - 1)$ then

$$\frac{u \cos(\theta) + x_0 w \sin(\theta)}{|V|^2 + (u \sin(\theta) + x_0 w \cos(\theta))^2} \geq C x_0.$$

which is clear when $x_0 \rightarrow 0$ because $u \sim 1$ and w is uniformly bounded.

Observe finally that if we have the singular case that the point of tangency $M = x_0 e_1 + (1 + \varepsilon)|x_0|^2$ of the upper paraboloid with the tangent hyperplane H is on ∂K , then we get a cap of K by pushing H a small distance into the upper paraboloid in the direction of its inner normal. \square

Lemma 8. *Under the assumptions of Lemma 5, if K is a local maximizer for L_K and ∂K has positive generalized curvature at X_0 then the outer normal $N(K, X_0)$ of K at X_0 is parallel to the vector X_0 .*

Proof: We assume that L_K is maximal, ∂K has positive generalized curvature at X_0 and the normal vector of K at X_0 is not parallel to X_0 .

Using Lemma 7 we continue as follows: Let $u \in S^{n-1}$ and $\alpha > 0$ be taken from Lemma 7. Let $H = \{X; \langle X, u \rangle = \alpha\}$ and $H^+ = \{X; \langle X, u \rangle \geq \alpha\}$. Let $M = \max\{|X|; X \in H^+ \cap K\}$. Then $M < |X_0|$. Let d be the distance from 0 to H , $h = h_K(u) - d$ and, for $m \geq 1$, let

$$D'_m = \{X \in K; h_K(u) - \frac{h}{m} \leq \langle X, u \rangle \leq h_K(u)\}.$$

Then the sequence D'_m satisfies the conditions of Lemma 2. We have

$$\int_{D'_m} |X|^2 \leq M^2 |D'_m|.$$

Now, since L_K is maximal, we have, combining the above with (3), for m big enough,

$$\frac{-M^2 |D'_m|}{M_K^2} + (n+2) \frac{|D'_m|}{|K|} \leq O(|D'_m|^2).$$

Combining the last inequality with (4) we get, passing to the limit as $m \rightarrow \infty$,

$$|X_0|^2 = \frac{(n+2)M_K^2}{|K|} \leq M^2 < |X_0|^2,$$

which is a contradiction. \square

3 Proof of Theorem 1.

Assume that K is a local maximizer of L_K and $X_0 \in \partial K$ is a point of positive generalized curvature of ∂K . We may assume that K is in isotropic position.

By Lemma 8, we know that $u = \frac{X_0}{|X_0|}$ is the external normal of K at X_0 . We choose for K_m and K'_m , $m \geq 1$, the following sets:

$$K_m = \text{conv}(X_0 + \frac{u}{m}, K)$$

and

$$K'_m = \{X \in K; \langle X, u \rangle \leq \langle X_0, u \rangle - \frac{1}{m}\}.$$

By Remark 2) following Lemma 5, the sets $K_m \setminus K$ and $K \setminus K'_m$ satisfy the conditions of Lemma 5 and, of course, of Proposition 4. In view of Lemma 5, it is essential to have an accurate estimation of

$$\int_{K_m \setminus K} |X|^2 dX - |X_0|^2 |K_m \setminus K| = \int_{K_m \setminus K} (|X|^2 - |X_0|^2) dX$$

and

$$\int_{K \setminus K'_m} |X|^2 dX - |X_0|^2 |K \setminus K'_m| = \int_{K \setminus K'_m} (|X|^2 - |X_0|^2) dX.$$

For having such estimation it would be convenient to assume that the standard approximating ellipsoid of K at X_0 is a Euclidean ball rather than just an ellipsoid.

Let u_1, \dots, u_n be an orthonormal system in \mathbb{R}^n , with $u_n = \frac{X_0}{|X_0|}$ and such that u_1, \dots, u_{n-1} are the directions of the principal radii of the quadratic form q associated with X_0 (see Definition 1). Let $T \in SL(n)$ be a volume preserving linear transformation of the form

$$T(\sum_{j=1}^n x_j u_j) = \sum_{j=1}^n \lambda_j x_j u_j; \quad \prod_{j=1}^n \lambda_j = 1$$

(we write in short $T(X) = \Lambda X$ and $T^{-1}(X) = \Lambda^{-1} X$ assuming X is written using the basis u_1, \dots, u_n). Choose T so that the standard approximating ellipsoid of $\tilde{K} = T(K)$ at $T(X_0)$ is a Euclidean ball of radius R .

Denoting $\tilde{K}_m = T(K_m)$ and $\tilde{K}'_m = T(K'_m)$ we get

$$\int_{K_m \setminus K} (|X|^2 - |X_0|^2) dX = \int_{\tilde{K}_m \setminus \tilde{K}} (|\Lambda^{-1} Y|^2 - |\Lambda^{-1} Y_0|^2) dY$$

and

$$\int_{K \setminus K'_m} (|X|^2 - |X_0|^2) dX = \int_{\tilde{K} \setminus \tilde{K}'_m} (|\Lambda^{-1} Y|^2 - |\Lambda^{-1} Y_0|^2) dY.$$

We shall use a temporary coordinate system that satisfies:

- 1) $T(X_0) = 0$

- 2) The outer normal vector of \tilde{K} at 0 is $-e_n$ (e_n is the n -th coordinate vector), thus $\tilde{K} \subset \{X \in \mathbb{R}^n; \langle X, e_n \rangle \geq 0\}$

We write $X = (Y, y) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. Let $G = g(\tilde{K})$ be the centroid of \tilde{K} . In our temporary coordinates $G = (0, b)$ with $b > 0$ (in view of Lemma 8). For $a > 0$, small enough, define

$$C_a = \text{conv}(\tilde{K}, (-a, 0)) \setminus \tilde{K}$$

$$D_a = \{(Y, y) \in \tilde{K}; y \leq a\}.$$

By the above discussion, we have to estimate for $\tilde{K}_m \setminus \tilde{K} = C_a$ and $\tilde{K} \setminus \tilde{K}'_m = D_a$ ($a = \frac{1}{m}$), the following quantities in terms of $a > 0$, $a \rightarrow 0$:

$$\phi(a) = \int_{C_a} (|\Lambda^{-1}(X - G)|^2 - |\Lambda^{-1}G|^2) dX$$

$$\psi(a) = \int_{D_a} (|\Lambda^{-1}(X - G)|^2 - |\Lambda^{-1}G|^2) dX.$$

The equation of the boundary of the body, in a neighborhood of 0 can be written as

$$y = \frac{|Y|^2}{2R} + o(|Y|^2),$$

With these notations

$$\phi(a) = \int_{(Y,y) \in C_a} \left(\sum_{j=1}^{n-1} \left(\frac{Y_j}{\lambda_j} \right)^2 + \left(\frac{y}{\lambda_n} \right)^2 - 2 \frac{yb}{\lambda_n^2} \right) dY dy,$$

$$\psi(a) = \int_{(Y,y) \in D_a} \left(\sum_{j=1}^{n-1} \left(\frac{Y_j}{\lambda_j} \right)^2 + \left(\frac{y}{\lambda_n} \right)^2 - 2 \frac{yb}{\lambda_n^2} \right) dY dy.$$

We first estimate $\phi(a)$ and $\psi(a)$ under the hypothesis that in some neighborhood of 0 the equation of the boundary of K is actually

$$y = \frac{|Y|^2}{2R}.$$

Then we shall see that this approximation is actually good.

- 1) We suppose that $y = \frac{|Y|^2}{2R}$. One has

$$D_a = \{(Y, y) \in \mathbb{R}^n; |Y| \leq \sqrt{2Ra}, \frac{|Y|^2}{2R} \leq y \leq a\}.$$

Since D_a is circular with respect to Y , we have

$$\int_{(Y,y) \in D_a} Y_j^2 dY dy = \frac{1}{n-1} \int_{(Y,y) \in D_a} |Y|^2 dY dy.$$

Substituting $\alpha_n = \frac{1}{n-1} \sum_{j=1}^{n-1} \lambda_j^{-1}$ we get with a change of variable to polar coordinates in \mathbb{R}^{n-1} and denoting by v_k the volume of the Euclidean ball in \mathbb{R}^k ,

$$\psi(a) = (n-1)v_{n-1} \int_{S_{n-2}} \int_0^{\sqrt{2Ra}} \left(\int_{\frac{r^2}{2R}}^a (\alpha_n r^2 + \lambda_n^{-1}(y^2 - 2yb)) dy \right) r^{n-2} dr d\theta.$$

Setting $r = \sqrt{2Ra}s$ and $y = az$ we get

$$\begin{aligned} \psi(a) &= (n-1)v_{n-1}a(2Ra)^{\frac{n-1}{2}} \int_{S_{n-2}} \int_0^1 \left(\int_{s^2}^1 (2\alpha_n Ras^2 + \lambda_n^{-1}(a^2z^2 - 2abz)) dz s^{n-2} ds d\theta \right) \\ &= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}} \int_0^1 (2\alpha_n Ras^n(1-s^2) + \lambda_n^{-1}(\frac{1}{3}a^2s^{n-2}(1-s^6) - ab(1-s^4)s^{n-2})) ds \\ &= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+3}{2}} \int_0^1 (2\alpha_n Rs^n(1-s^2) + \lambda_n^{-1}(\frac{a}{3}s^{n-2}(1-s^6) - b(1-s^4)s^{n-2})) ds \\ &= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+3}{2}} \left((2\alpha_n R(\frac{1}{n+1} - \frac{1}{n+3}) - \lambda_n^{-1}b(\frac{1}{n-1} - \frac{1}{n+3}) + O(a) \right) \\ &= 4(n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+3}{2}} \left(\frac{\alpha_n R}{(n+1)(n+3)} - \frac{\lambda_n^{-1}b}{(n-1)(n+3)} + O(a) \right) \\ &= \frac{4(n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+3}{2}}}{(n+1)(n+3)} \cdot \left(\alpha_n R - \frac{(n+1)\lambda_n^{-1}b}{n-1} + O(a) \right). \end{aligned}$$

We shall need also to compute $|D_a|$. One has

$$\begin{aligned} |D_a| &= (n-1)v_{n-1}a(2Ra)^{\frac{n-1}{2}} \int_{S_{n-2}} \int_0^1 (1-s^2)s^{n-2} ds d\theta \\ &= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) = \frac{2v_{n-1}}{n+1} (2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}}. \end{aligned}$$

2) We still suppose that the boundary of \tilde{K} in a neighborhood of 0 is given by $y = \frac{|Y|^2}{2R}$. Then the tangent hyperplanes to \tilde{K} through $(0, -a)$, indexed by $\theta \in S^{n-2}$ - the direction of the projection of their point of tangency with \tilde{K} , are given by the equations

$$y = -a + \sqrt{\frac{2a}{R}} \langle \theta, Y \rangle.$$

It follows that

$$\begin{aligned} C_a &= \left\{ (Y, y) \in \mathbb{R}^n; |Y| \leq \sqrt{2Ra}, -a + \sqrt{\frac{2a}{R}}|Y| \leq y \leq \frac{|Y|^2}{2R} \right\} \\ &= \left\{ (\sqrt{2Ra}Z, az) \in \mathbb{R}^n; |Z| \leq 1, 2|Z| - 1 \leq z \leq |Z|^2 \right\}. \end{aligned}$$

Thus, using the same rotation invariance as in (1),

$$\phi(a) = (n-1)v_{n-1}a(2Ra)^{\frac{n-1}{2}} \int_{S_{n-2}} \int_0^1 \left(\int_{2s-1}^{s^2} (2\alpha_n Ras^2 + \lambda_n^{-1}(a^2z^2 - 2abz)) s^{n-2} ds \right) d\theta$$

$$\begin{aligned}
&= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}}a^{\frac{n+3}{2}}\left(\int_0^1((2s^n(1-s)^2\alpha_n R - \lambda_n^{-1}b(s^4 - (2s-1)^2)s^{n-2})ds + O(a)\right) \\
&= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}}a^{\frac{n+3}{2}}\left(\left(\frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3}\right)2\alpha_n R - \left(\frac{1}{n+3} - \frac{4}{n+1} + \frac{4}{n} - \frac{1}{n-1}\right)\lambda_n^{-1}b + O(a)\right) \\
&= (n-1)v_{n-1}(2R)^{\frac{n-1}{2}}a^{\frac{n+3}{2}}\left(\frac{4\alpha_n R}{(n+1)(n+2)(n+3)} - \frac{4(n-3)\lambda_n^{-1}b}{n(n+1)(n-1)(n+3)} + O(a)\right) \\
&= \frac{4(n-1)v_{n-1}(2R)^{\frac{n-1}{2}}a^{\frac{n+3}{2}}}{(n+1)(n+3)} \cdot \left(\frac{\alpha_n R}{n+2} - \frac{(n-3)\lambda_n^{-1}b}{n(n-1)} + O(a)\right).
\end{aligned}$$

Moreover

$$|C_a| = (n-1)v_{n-1}a(2Ra)^{\frac{n-1}{2}} \int_{S_{n-2}} \left(\int_0^1 (1-s)^2 s^{n-2} ds\right) d\theta = \frac{2v_{n-1}}{n(n+1)} (2R)^{\frac{n-1}{2}} a^{\frac{n+1}{2}}.$$

3) But the hypothesis which has been done that in a neighborhood of 0, the equation of the boundary of \tilde{K} is $y = \frac{|Y|^2}{2R}$ has to be replaced with the following one: For every $\varepsilon > 0$, there exists $c \geq 0$ such that

$$(1-\varepsilon)\frac{|Y|^2}{2R} \leq y \leq (1+\varepsilon)\frac{|Y|^2}{2R} \text{ whenever } |Y|^2 + y^2 \leq c.$$

One has to see that in terms of a , the estimates of **2)** and **3)** still hold. We shall treat first $\psi(a)$ and then $\phi(a)$.

One has

$$D_a \subset \{(Y, y) \in \mathbb{R}^n; |Y| \leq \sqrt{2R_+(a)a}, \frac{|Y|^2}{2R_+(a)} \leq y \leq a\},$$

$$\{(Y, y) \in \mathbb{R}^n; |Y| \leq \sqrt{2R_-(a)a}, \frac{|Y|^2}{2R_-(a)} \leq y \leq a\} \subset D_a$$

and

$$C_a \subset \{(Y, y) \in \mathbb{R}^n; |Y| \leq \sqrt{2R_+(a)a}, -a + \sqrt{\frac{2a}{R_+(a)}} \leq y \leq \frac{|Y|^2}{2R_-(a)}\},$$

$$\{(Y, y) \in \mathbb{R}^n; |Y| \leq \sqrt{2R_-(a)a}, -a + \sqrt{\frac{2a}{R_-(a)}} \leq y \leq \frac{|Y|^2}{2R_+(a)}\} \subset C_a,$$

with $R_+(a) = R + \varepsilon_+(a)$ and $R_-(a) = R - \varepsilon_-(a)$, where $\varepsilon_+(a)$ and $\varepsilon_-(a)$ are nonnegative functions tending to 0 when $a \rightarrow 0$. Then everything works with upper and lower bounds for the negative and the positive terms on D_a and C_a , observing also that that $|D_a|^2$ and $|C_a|^2$ are of the order of a^{n+1} which is negligible with respect to $a^{\frac{n+3}{2}}$, so that we can apply Proposition 4.

Remark. The importance of Lemma 8 comes in step **3)** above. Here, if the normal vector of \tilde{K} at 0 were not parallel to the y -axis, we would get an extra error term of order that could be estimated only by $a^{\frac{n+2}{2}}o(a)$. For our proof of Theorem 1 to work we would need an estimate of order $a^{\frac{n+3}{2}}o(a)$ for this term.

To conclude, using Proposition 4 and Lemma 5 (including (5) in its proof) and replacing $K_m \setminus K$ by $T^{-1}(C_a)$ and $K \setminus K'_m$ by $T^{-1}(D_a)$, the above computations show that for some functions $c(n, R)$ and $d(n, R)$ depending only of n and R ,

$$L_{K_m}^{2n} = L_K^{2n} \left(1 + c(n, R) a^{\frac{n+3}{2}} \left(\alpha_n R - \frac{(n+2)(n-3)}{n(n-1)} \lambda_n^{-1} b + O(a) \right) \right)$$

and

$$L_{K'_m}^{2n} = L_K^{2n} \left(1 - d(n, R) a^{\frac{n+3}{2}} \left(\alpha_n R - \frac{n+1}{n-1} \lambda_n^{-1} b + O(a) \right) \right).$$

Thus one has both

$$\alpha_n \lambda_n R \leq \frac{(n+2)(n-3)}{n(n-1)} b \quad \text{and} \quad \alpha_n \lambda_n R \geq \frac{n+1}{n-1} b,$$

So that

$$\frac{(n+2)(n-3)}{n(n-1)} \geq \frac{n+1}{n-1}$$

which gives a contradiction.

Note that in the case that K is centrally symmetric, a similar argument, using C_m and $-C_m$ together and D_m and $-D_m$ together will work in the same way, keeping K_m and K'_m centrally symmetric. This observation takes care of the centrally symmetric part of Theorem 1. There the use of lemma 3 is not needed, due to symmetry. \square

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